Introduction to
SET

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The Game:

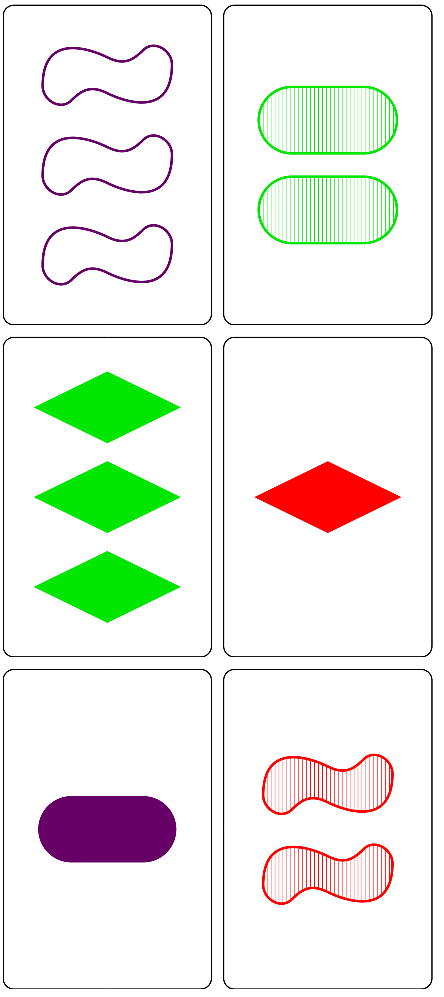
SET® is a card game for one or more players, played with a special deck of cards.

Each SET card has four attributes:

number, color, shading, and shape.

On a given card, each attribute takes on one of three values:

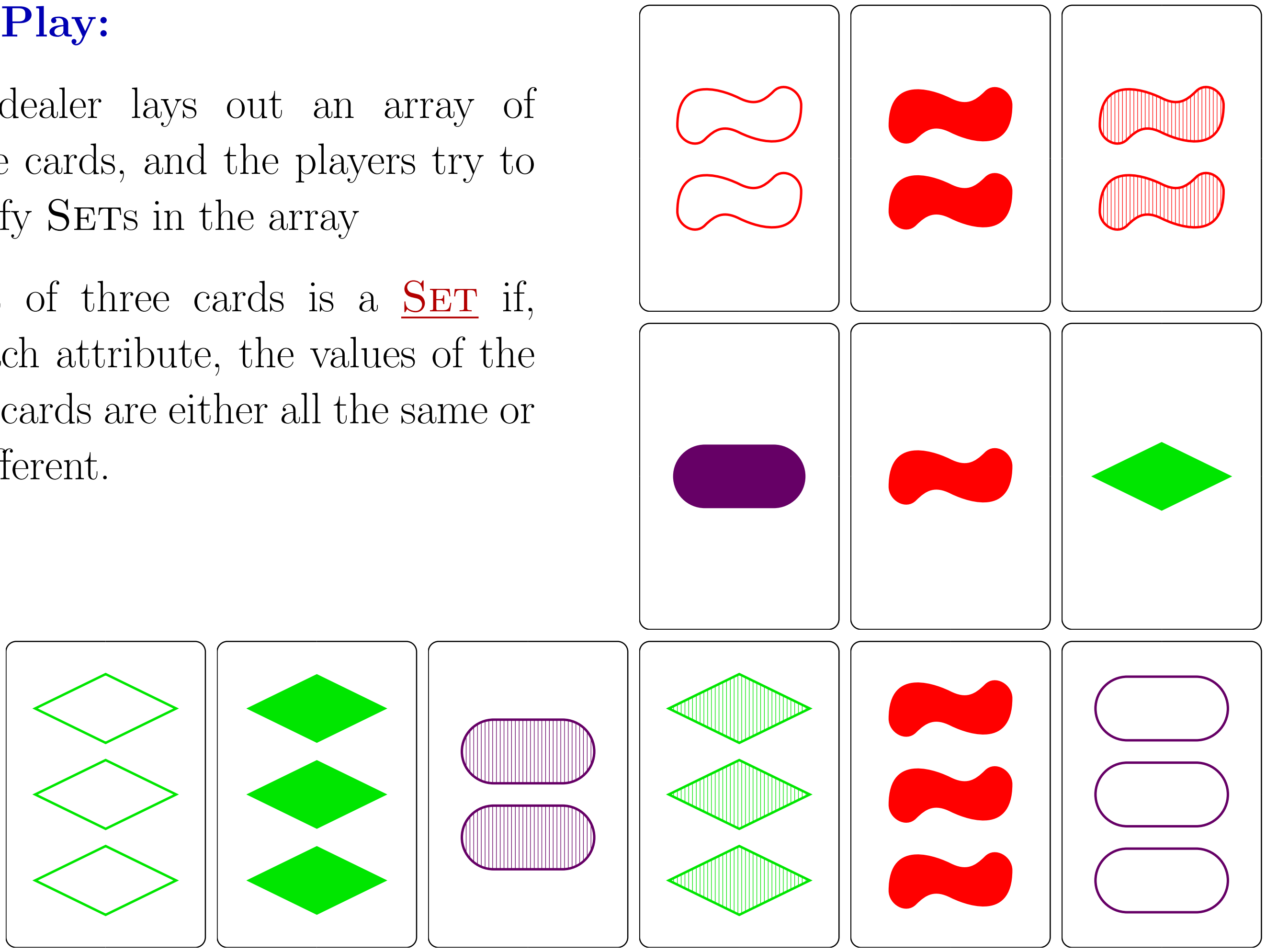
number	color	shading	shape
1	red	filled	diamond
2	green	outlined	oval
3	purple	striped	squiggle



The Play:

The dealer lays out an array of twelve cards, and the players try to identify SETs in the array

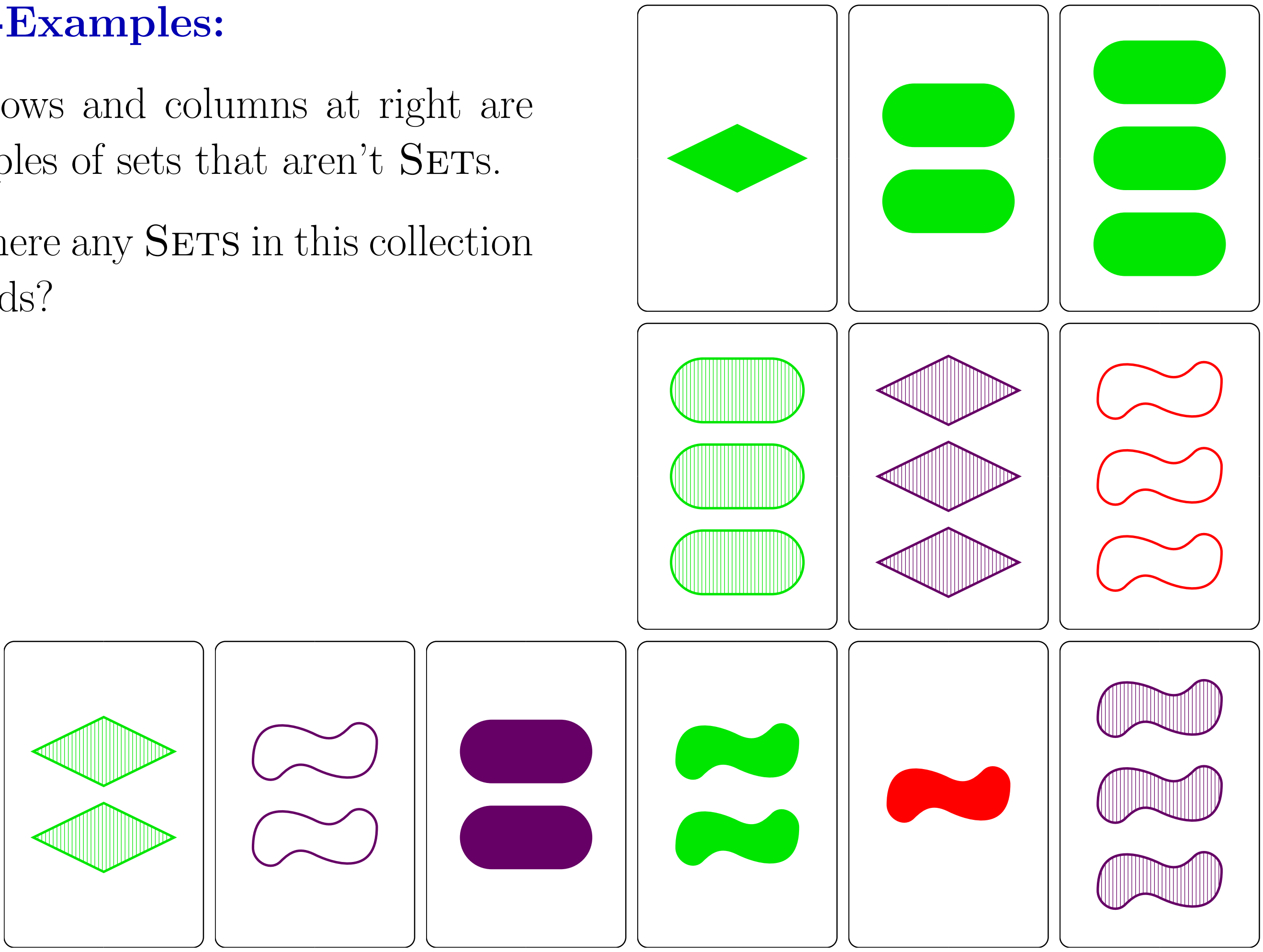
A set of three cards is a SET if, for each attribute, the values of the three cards are either all the same or all different.



Non-Examples:

The rows and columns at right are examples of sets that aren't SETS.

Are there any SETS in this collection of cards?

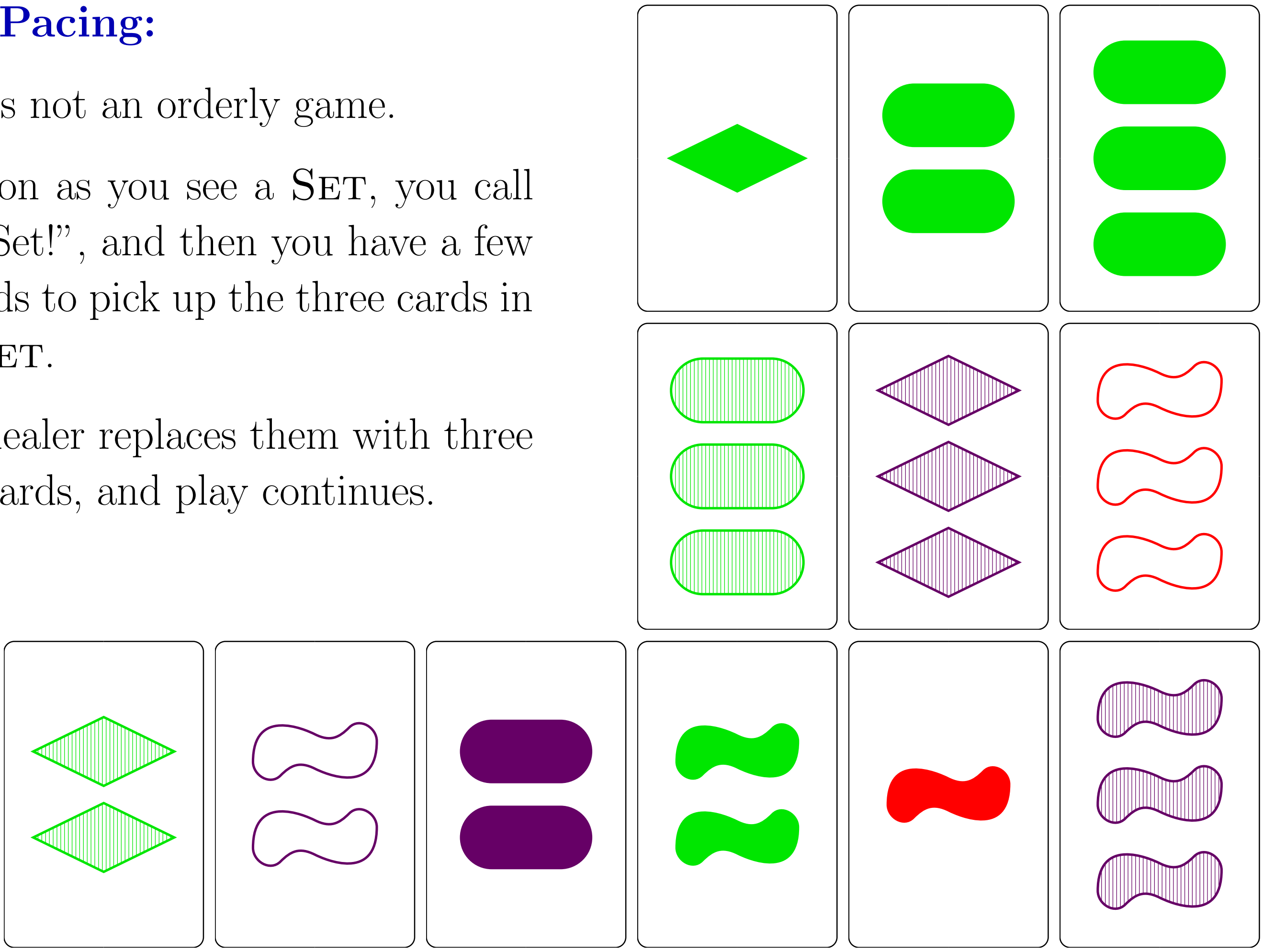


The Pacing:

This is not an orderly game.

As soon as you see a Set, you call out "Set!", and then you have a few seconds to pick up the three cards in the Set.

The dealer replaces them with three new cards, and play continues.



Observations:

- With three possible values for each of four attributes, there can be $3 \times 3 \times 3 \times 3 = 81$ different cards, and in fact the deck contains just these 81 cards.
- Every pair of cards determines exactly one SET. That is, given cards c_1 and c_2, there is a unique card c_3 such that $\{c_1, c_2, c_3\}$ is a SET.

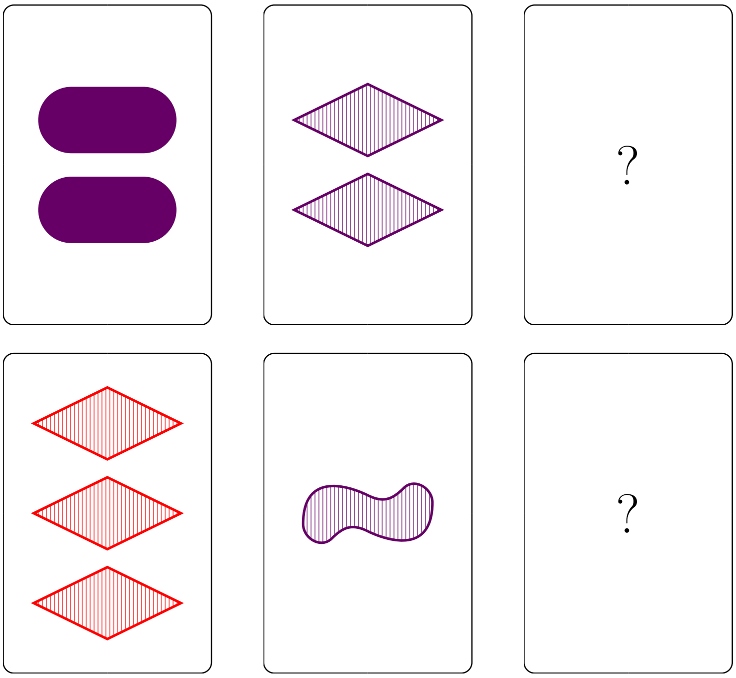
  Proof: For each attribute, c_1 and c_2 either have the same value or different values.

  If they have the same value, then c_3 must also have that value.

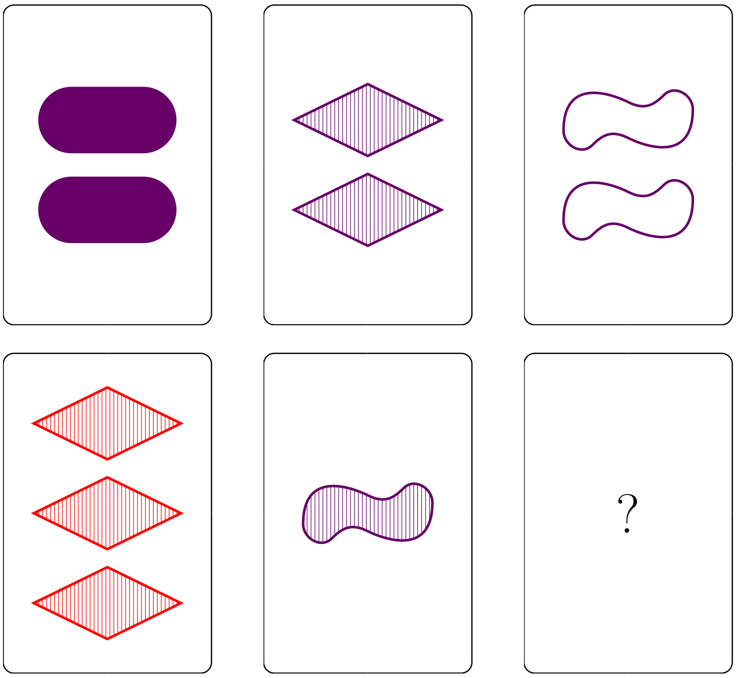
  If they have different values, then c_3 must have the third possible value for that attribute.

  Thus the value of each of c_3's attributes is determined by c_1 and c_2.

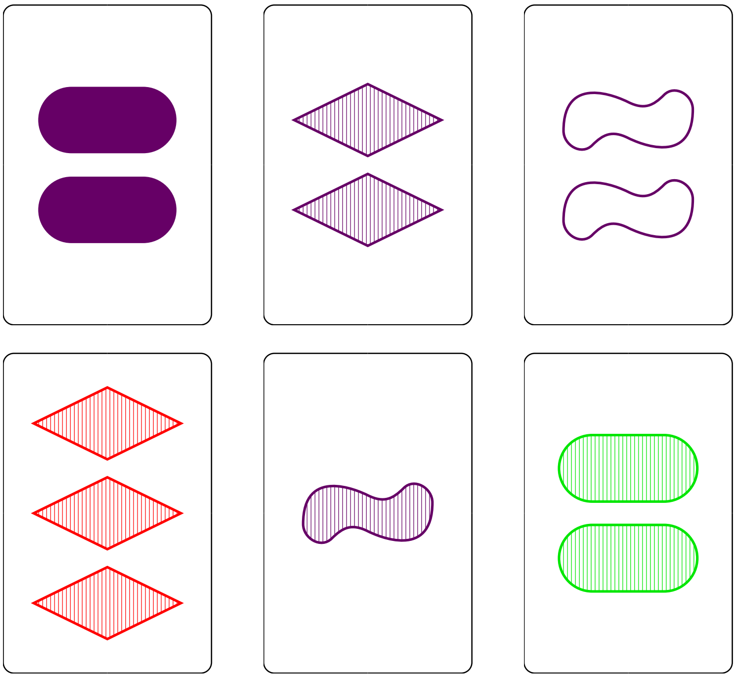
Examples:



Examples:

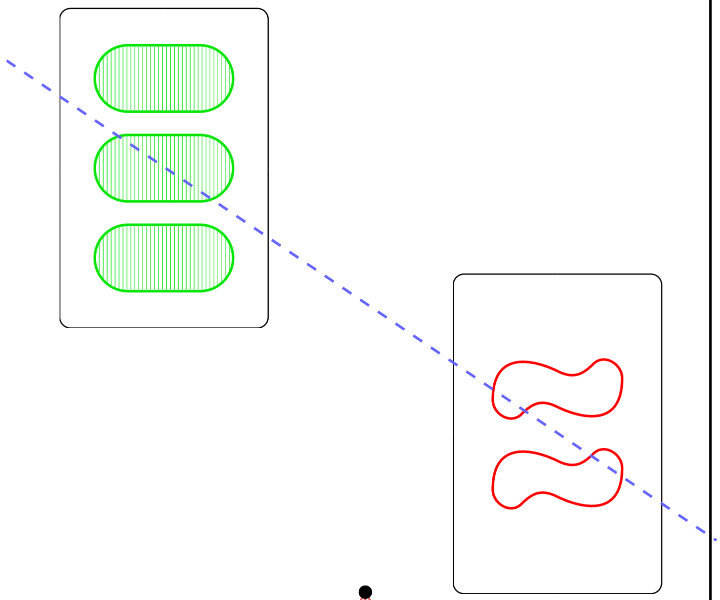


Examples:



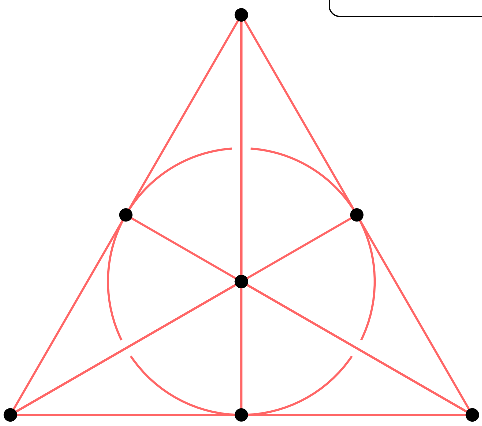
Analogy:

Two cards determine a SET
just as
two points determine a line.



Closer Analogy:

In the Fano Plane, every line is determined by two points and every line contains exactly three points.



Modelling the Deck:

Map the three possible values of each attribute onto the set $\{0, 1, 2\}$ in a bijective, but otherwise arbitrary, way.

number			color			shading			shape		
3	$\leftrightarrow$	0	red	$\leftrightarrow$	0	filled	$\leftrightarrow$	0	diamond	$\leftrightarrow$	0
1	$\leftrightarrow$	1	green	$\leftrightarrow$	1	outlined	$\leftrightarrow$	1	oval	$\leftrightarrow$	1
2	$\leftrightarrow$	2	purple	$\leftrightarrow$	2	striped	$\leftrightarrow$	2	squiggle	$\leftrightarrow$	2

This gives us a one-to-one mapping from the SET deck onto $(\mathbb{F}_3)^4$.

We identify each card with an ordered quadruple or four-dimensional vector:

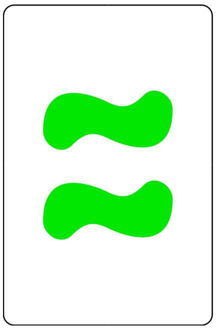
$$\mathbf{x} = (x_1, x_2, x_3, x_4)$$

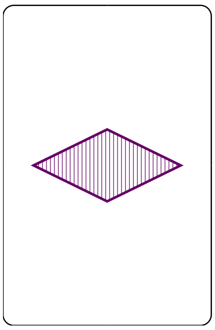
with $x_i \in \mathbb{F}_3$ for each i.

Examples:

number			color			shading			shape		
3	$\leftrightarrow$	0	red	$\leftrightarrow$	0	filled	$\leftrightarrow$	0	diamond	$\leftrightarrow$	0
1	$\leftrightarrow$	1	green	$\leftrightarrow$	1	outlined	$\leftrightarrow$	1	oval	$\leftrightarrow$	1
2	$\leftrightarrow$	2	purple	$\leftrightarrow$	2	striped	$\leftrightarrow$	2	squiggle	$\leftrightarrow$	2

The quadruple $\mathbf{x} = (2, 1, 0, 2)$
denotes two green filled squiggles.





The quadruple $\mathbf{y} = (1, 2, 2, 0)$
denotes one purple striped diamond.

Applying the model:

The rules of the game say that the vectors

$$(x_1, x_2, x_3, x_4), \quad (y_1, y_2, y_3, y_4), \quad \text{and} \quad (z_1, z_2, z_3, z_4)$$

form a SET if and only if, for each $i \in \{1, 2, 3, 4\}$,

$$\text{either } x_i = y_i = z_i \text{ or } \{x_i, y_i, z_i\} = \{0, 1, 2\}$$

Claim: Let x, y, and z be elements of $\mathbb{F}_3$. Then

$$\boxed{\begin{array}{c} x = y = z \\ \text{or} \\ \{x, y, z\} = \{0, 1, 2\} \end{array}} \iff \boxed{x + y + z \equiv 0 \pmod 3}$$

Claim: Let x, y, and z be elements of $\mathbb{F}_3$. Then

$$\boxed{\begin{array}{c} x = y = z \\ \text{or} \\ \{x, y, z\} = \{0, 1, 2\} \end{array}} \iff \boxed{x + y + z \equiv 0 \pmod 3}$$

Proof:

($\Rightarrow$) If $x = y = z$, then $x + y + z = 3x \equiv 0 \pmod 3$.
If $\{x, y, z\} = \{0, 1, 2\}$, then $x + y + z = 3 \equiv 0 \pmod 3$.

($\Leftarrow$) Suppose $x + y + z \equiv 0 \pmod 3$ and $\{x, y, z\} \neq \{0, 1, 2\}$. Then two of x, y, and z are equal, and we may assume without loss of generality that $y = x$. Then $x + y + z = 2x + z$ and we have

$$\begin{array}{rcrcll} 2x & + & z & \equiv & 0 & \pmod 3 \\ x & & & \equiv & x & \pmod 3 \\ \hline & & z & \equiv & x & \pmod 3 \end{array}$$

This implies that $z = x$, so we have $x = y = z$. ■

Corollary: Three cards

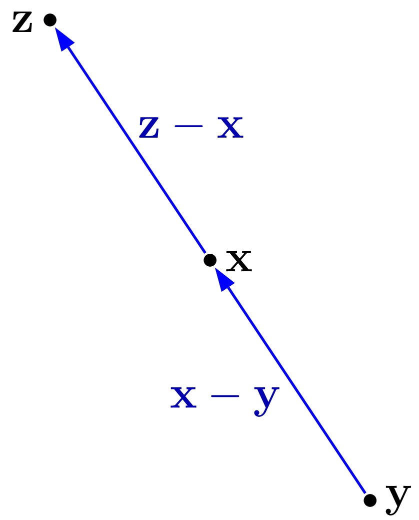
$$\mathbf{x} = (x_1, x_2, x_3, x_4), \quad \mathbf{y} = (y_1, y_2, y_3, y_4), \quad \text{and} \quad \mathbf{z} = (z_1, z_2, z_3, z_4)$$

form a SET if and only if

$$\mathbf{x} + \mathbf{y} + \mathbf{z} \ = \ \mathbf{0} \text{ in } (\mathbb{F}_3)^4.$$

Furthermore, in $(\mathbb{F}_3)^4$, we have

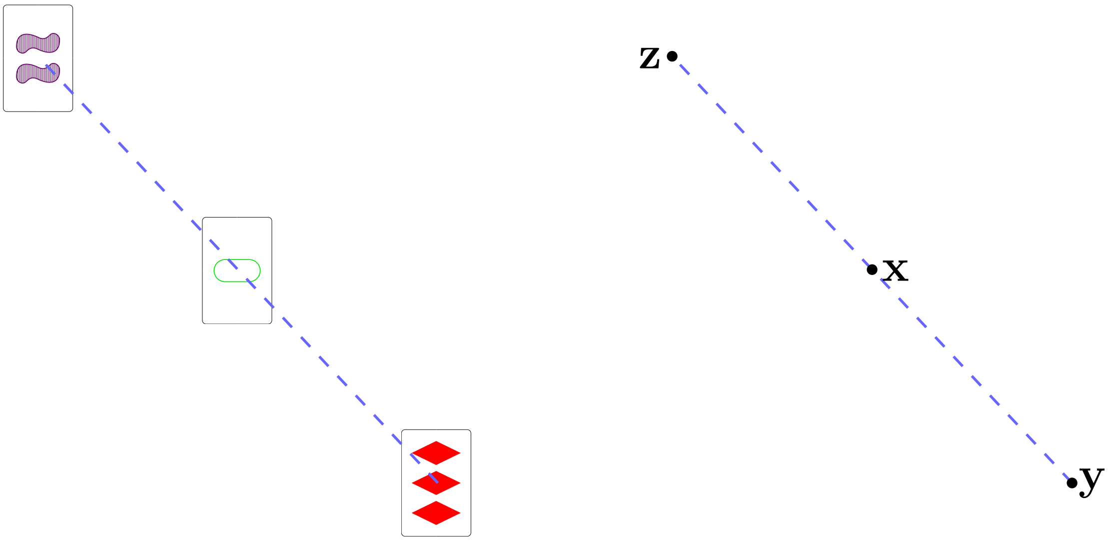
$$\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$$
$$\Longleftrightarrow$$
$$\mathbf{z} - \mathbf{x} = -2\mathbf{x} - \mathbf{y}$$
$$\Longleftrightarrow$$
$$\mathbf{z} - \mathbf{x} = \mathbf{x} - \mathbf{y}$$
$$\Longleftrightarrow$$
$\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are collinear



Punch Line:

Since every line in $(\mathbb{F}_3)^4$ contains exactly three points, we get

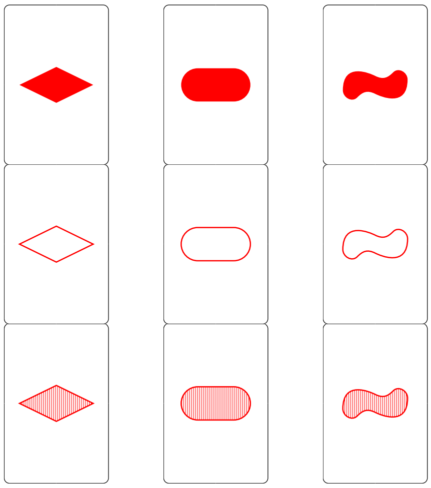
Corollary: Cards $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ form a SET if and only if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a line in $(\mathbb{F}_3)^4$.



Simplification:

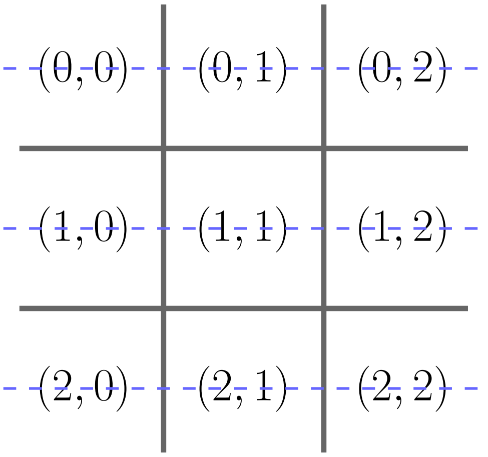
It's easier to picture two dimensions. Here's a picture of $(\mathbb{F}_3)^2$. This is a model for a two-attribute SET game.

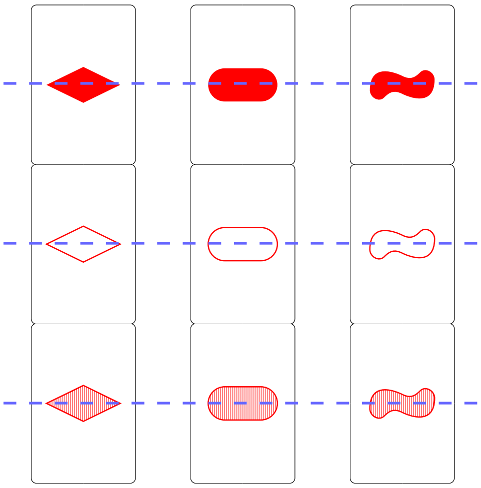
$(0,0)$	$(0,1)$	$(0,2)$
$(1,0)$	$(1,1)$	$(1,2)$
$(2,0)$	$(2,1)$	$(2,2)$



Simplification:

Here are some lines in $(\mathbb{F}_3)^2$.

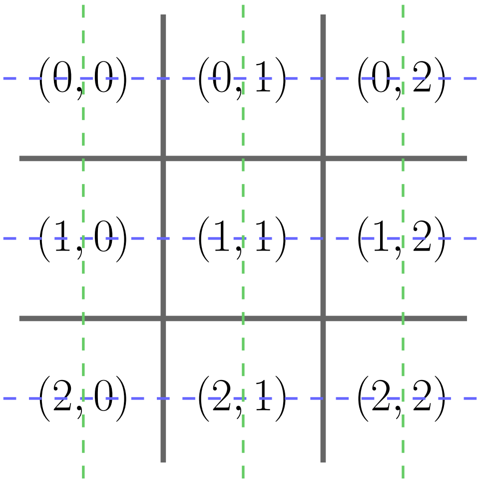


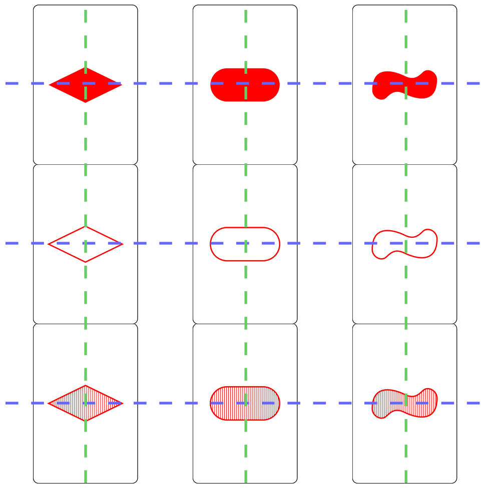


Simplification:

Here are some lines in $(\mathbb{F}_3)^2$.

Here are some more.



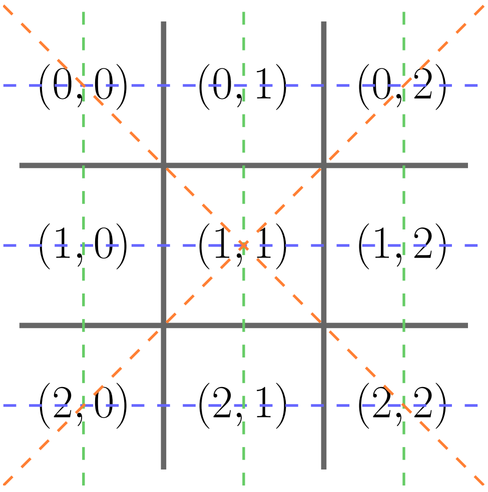


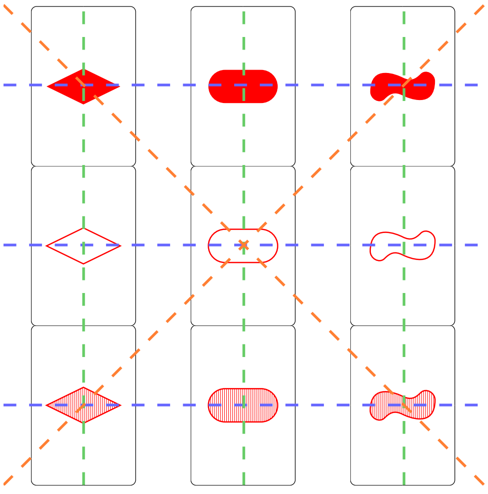
Simplification:

Here are some lines in $(\mathbb{F}_3)^2$.

Here are some more.

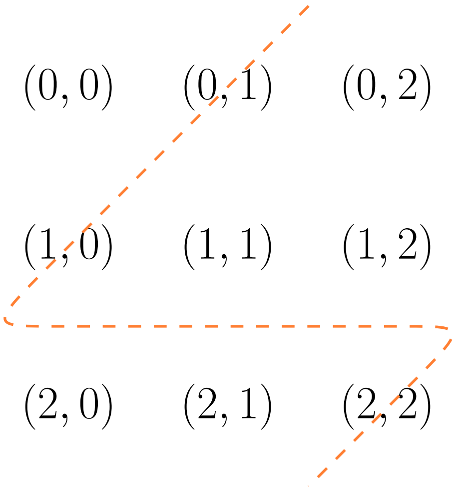
And more.

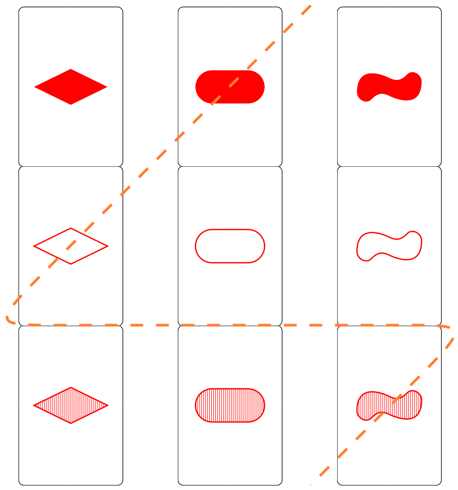




Simplification:

There are still other lines that “wrap around”.

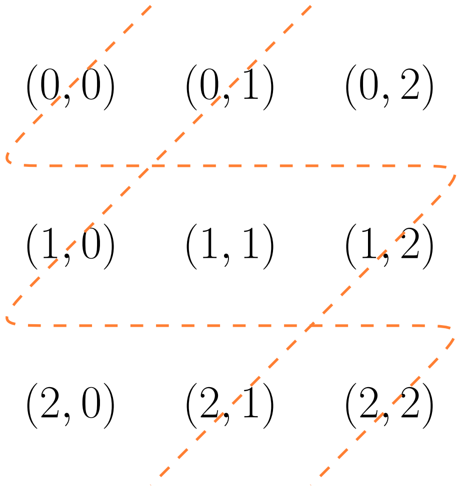


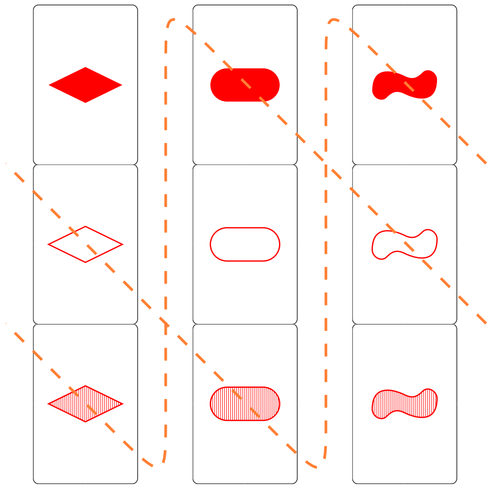


Simplification:

There are still other lines that "wrap around".

There are four of these.

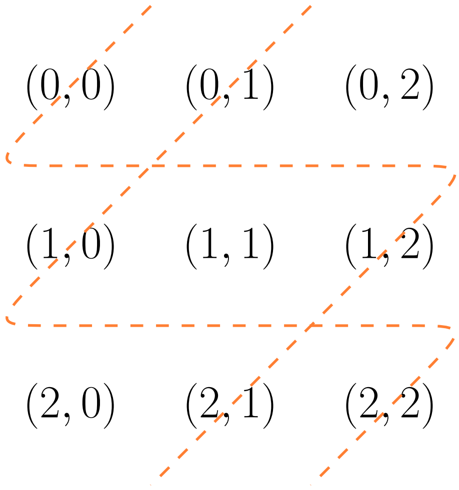


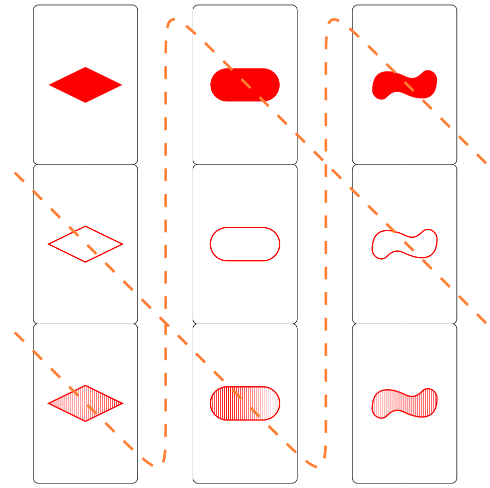


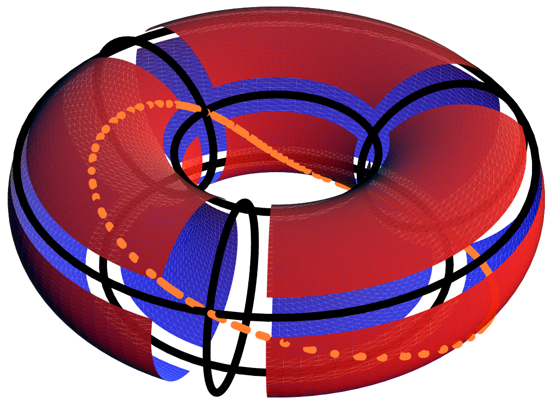
Simplification:

There are still other lines that "wrap around".

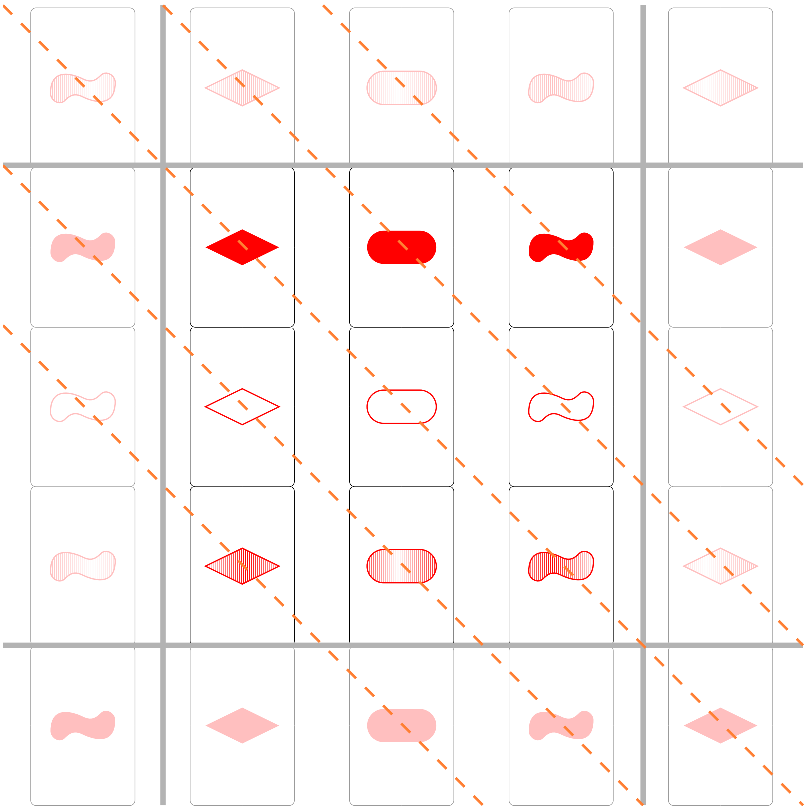
There are four of these.



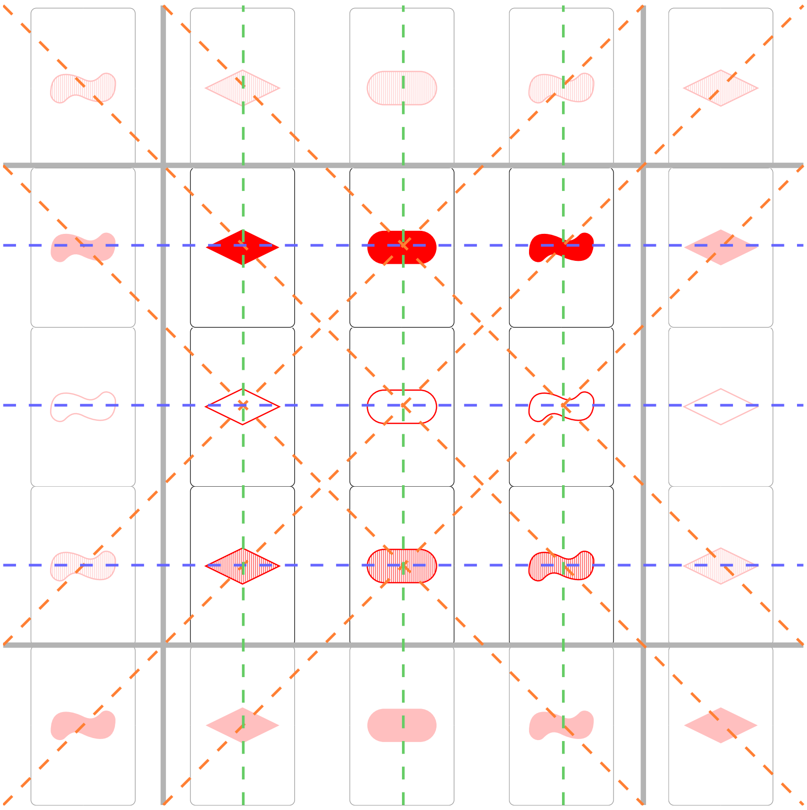




Observation: If you put $(\mathbb{F}_3)^2$ on a torus, then the wrap-around lines look just line the ordinary diagonals.

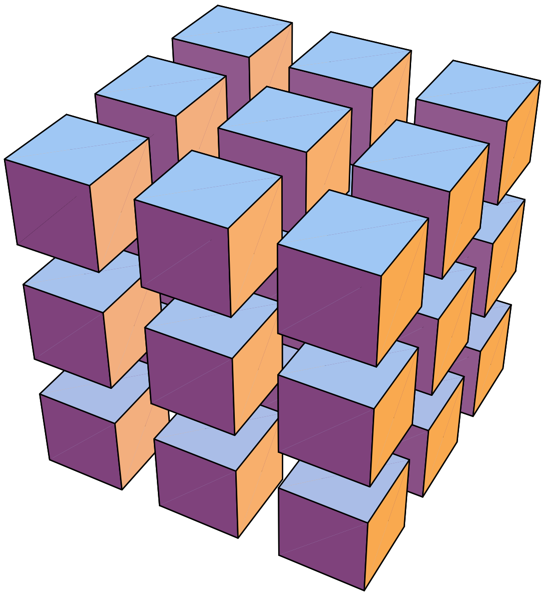


Summary: Here are all the lines in a two-attribute SET game.



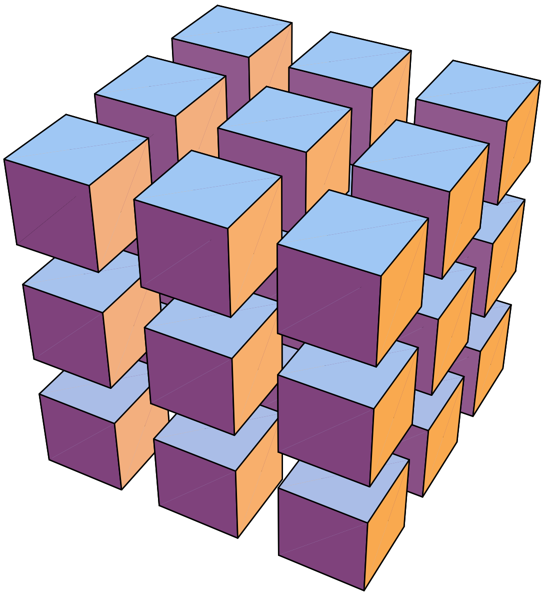
Extension:

The space $(\mathbb{F}_3)^3$ (a three-attribute SET game) looks like three-dimensional tic-tac-toe. To straighten out all the wrap-around lines, you'd identify opposite faces to put this array on a 3-torus.



Extension:

The space $(\mathbb{F}_3)^3$ (a three-attribute SET game) looks like three-dimensional tic-tac-toe. To straighten out all the wrap-around lines, you'd identify opposite faces to put this array on a 3-torus.



To visualize $(\mathbb{F}_3)^4$ (the full four-attribute SET game), go up one more dimension. We're looking for lines in an array of 81 hypercubes on a 4-torus.

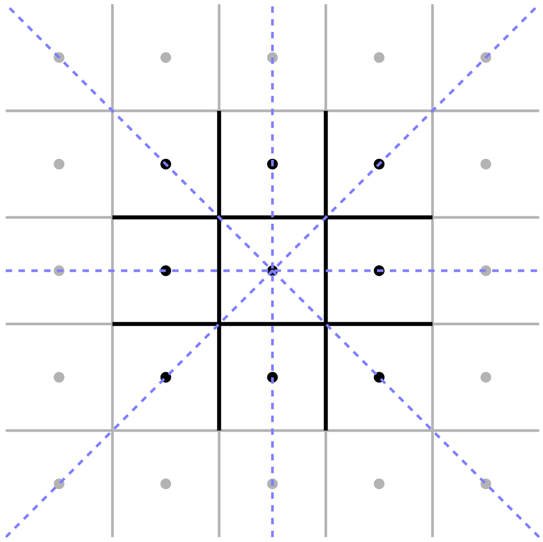
Easy Combinatorial Question: How many SETs are there? That is, how many lines are there in $(\mathbb{F}_3)^4$?

Solution I: Consider the set $\mathcal{I} = \{(\{\mathbf{x}, \mathbf{y}\}, \ell) \ : \ \mathbf{x} \text{ and } \mathbf{y} \text{ are on } \ell\}$.

- Each pair $\{\mathbf{x}, \mathbf{y}\}$ is on exactly one line, so
$$|\mathcal{I}| = \#(\text{pairs of points}) = \binom{81}{2}.$$
- Each line contains 3 points, so it contains $\binom{3}{2} = 3$ pairs of points, so
$$|\mathcal{I}| = 3 \times \#(\text{lines}).$$

Since $|\mathcal{I}|$ must equal $|\mathcal{I}|$, we get $3 \times \#(\text{lines}) = \binom{81}{2}$, so the number of lines in $(\mathbb{F}_3)^4$ is $\frac{1}{3}\binom{81}{2} = \boxed{1080}$.

Solution II (warm-up):



In $(\mathbb{F}_3)^2$, the two-attribute Set game, there are just four lines (in the torus picture) through each point:

2 for the pairs of opposite vertices ($V = 4$)
2 for the pairs of opposite edges ($E = 4$)

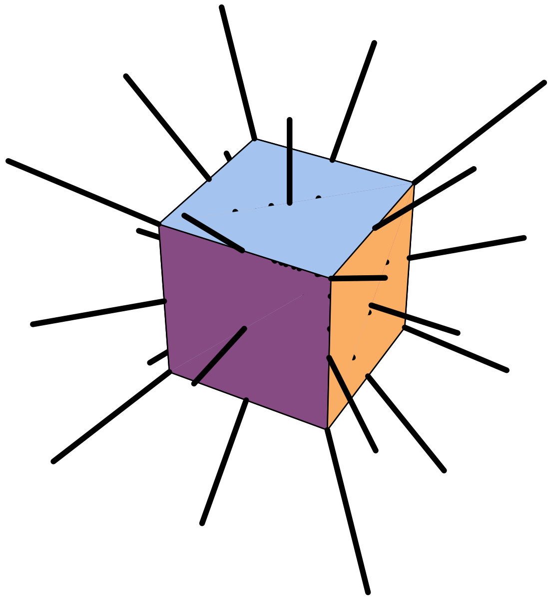
The total number of line-point incidences is

$$\#(\text{lines/point}) \times \#(\text{points}) = 4 \times 9 = 36$$

Each line accounts for 3 such incidences, so we get

$$\#(\text{lines}) = \frac{\#(\text{lines/point}) \times \#(\text{points})}{\#(\text{points/line})} = \frac{4 \times 9}{3} = 12$$

Solution II (phase 2):



In $(\mathbb{F}_3)^3$, the three-attribute SET game, each point lies on 13 lines in the 3-torus:

4 for the pairs of opposite vertices ($V = 8$)
6 for the pairs of opposite edges ($E = 12$)
3 for the pairs of opposite faces ($F = 6$)

We get

$$\frac{V + E + F}{2} = 13 \text{ lines/point.}$$

As before, the total number of lines in $(\mathbb{F}_3)^3$ is given by

$$\#(\text{lines}) = \frac{\#(\text{lines/point}) \times \#(\text{points})}{\#(\text{points/line})} = \frac{13 \times 27}{3} = 117$$

Solution II (continued):

We can't draw a hypercube in a 4-torus, but we can count the number of lines through each point of $(\mathbb{F}_3)^4$.

	vertices	edges	faces	cubes
segment	2	1	-	-
square	4	4	1	-
cube	8	12	6	1
hypercube	16	32	24	8

We get

$$\begin{array}{rl} 8 & \text{for the pairs of opposite vertices } (V = 16) \\ 16 & \text{for the pairs of opposite edges } (E = 32) \\ 12 & \text{for the pairs of opposite faces } (F = 24) \\ 4 & \text{for the pairs of opposite cubes } (C = 8) \end{array}$$

so there are $\dfrac{V+E+F+C}{2} = 40$ lines on the 4-torus through each point of $(\mathbb{F}_3)^4$

Solution II (conclusion):

The total number of lines in $(\mathbb{F}_3)^4$, the full four-attribute SET game, is

$$\frac{\#(\text{lines/point}) \times \#(\text{points})}{\#(\text{points/line})} = \frac{40 \times 81}{3} = 1080$$

as expected.

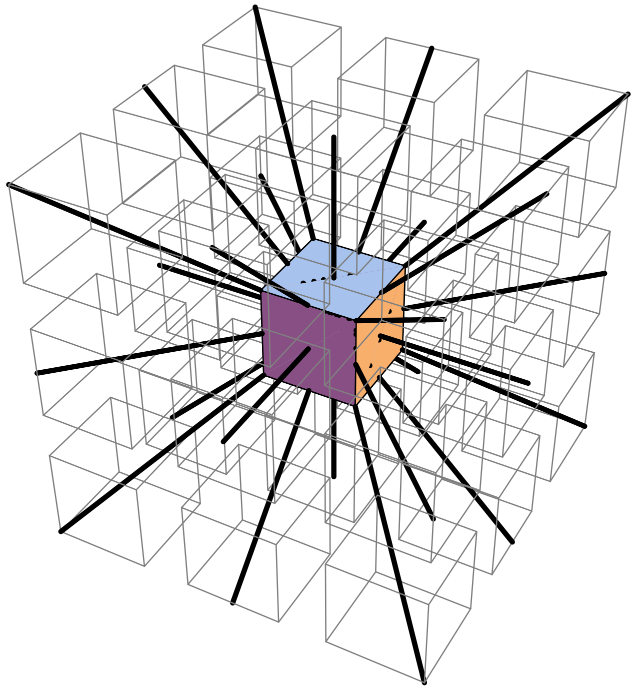
Solution II (dividend):

In $(\mathbb{F}_3)^3$, the three-attribute SET game,

the points on a "face" line have
2 attributes the same
and 1 different;

the points on an "edge" line have
1 attribute the same
and 2 different;

the points on a "vertex" line have
all 3 attributes different.



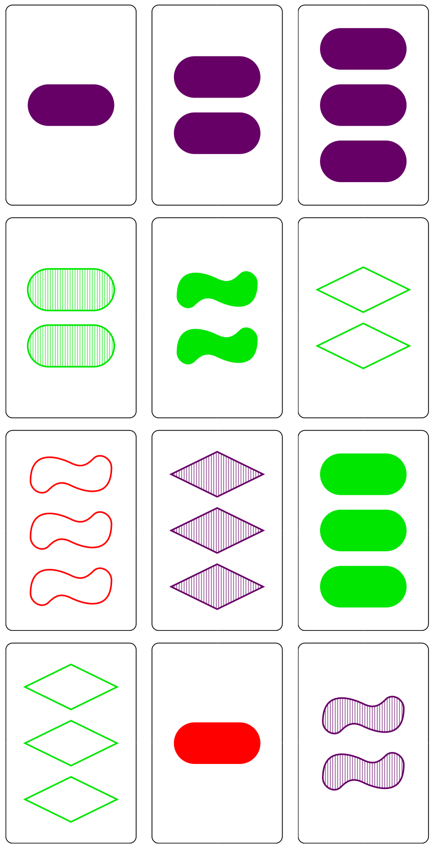
Solution II (dividend): If $(\mathbb{F}_3)^4$, there are four kinds of lines:

"cube" lines with
3 attributes the same
and 1 different;

"face" lines with
2 attributes the same
and 2 different;

"edge" lines with
1 attribute the same
and 3 different; and

"vertex" lines with
all 4 attributes different.



(Easy)

(Less easy)

(Tricky)

(Obscure)

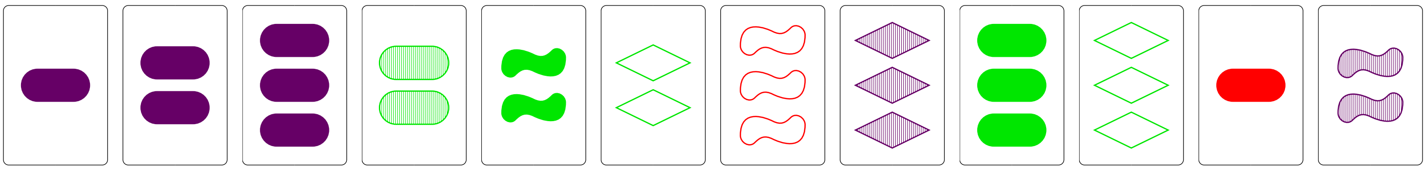
Solution II (dividend): We can partition the SETs by difficulty level:

We use the formula $\#(\text{lines}) = \dfrac{\#(\text{lines/point}) \times \#(\text{points})}{\#(\text{points/line})}$ and the fact that a hypercube has

8 cubes, 24 faces, 32 edges, and 16 vertices,

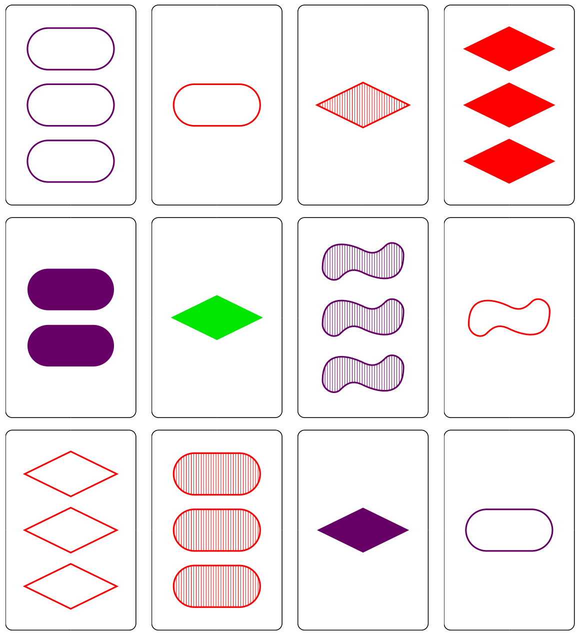
to get

$$\begin{array}{llllcrcr}
\# \text{ SETs with} & 3 \text{ same} & 1 \text{ different} & = & (8/2) \times 27 & = & 108 \\
\# \text{ SETs with} & 2 \text{ same} & 2 \text{ different} & = & (24/2) \times 27 & = & 324 \\
\# \text{ SETs with} & 1 \text{ same} & 3 \text{ different} & = & (32/2) \times 27 & = & 432 \\
\# \text{ SETs with} & 0 \text{ same} & 4 \text{ different} & = & (16/2) \times 27 & = & \underline{216} \\
 & & & & & & 1080
\end{array}$$



Harder combinatorial questions:

The instructions that come with the game say that if all players agree that an array of 12 cards contains no SET, then the dealer lays down three more cards and play continues.



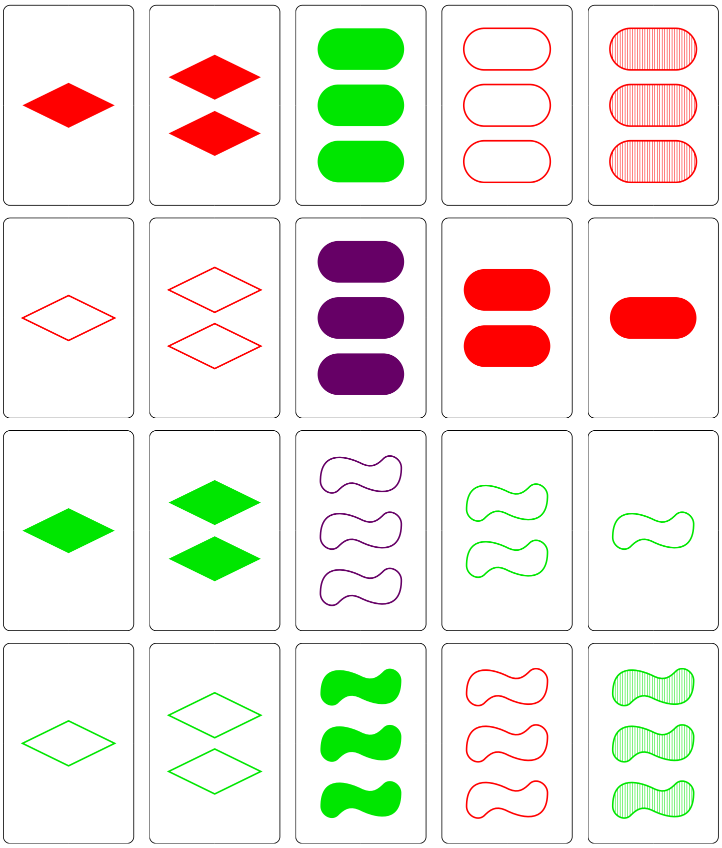
So . . .

- Does every collection of 15 cards contain a SET?
- If not, what is the smallest N such that every collection of N cards must contain a SET?

Some answers:

Here is a collection of 20 cards with no SET.

It has been shown (Pellegrino 1971, Davis and Maclagan 2003) that any collection of 21 cards contains a SET.



Even harder questions:

Again, according to the instructions,

$$P(\text{no } \textsc{Set} \text{ in 12 cards}) \approx 1/33 \quad \text{and} \quad P(\text{no } \textsc{Set} \text{ in 15 cards}) \approx 1/2500$$

Along these lines . . .

- For each N, how many of the $\binom{81}{N}$ collections of N cards are SET-free?
- For each N and k, how many of the $\binom{81}{N}$ collections of N cards contain exactly k sets?
- How does $P(\text{no } \textsc{Set} \text{ in 12 cards})$ change as play progresses? Are there any strategies better than speed and greed?

References:

- Robert Bosch, " 'Set'less collections of SET cards", *Optima* 63, 1999.
- Benjamin Lent Davis and Diane Maclagan, "The Card Game SET", *Mathematical Intelligencer* 25(3), 2003.